CCFU Proof 17

Two Canonical Four-Mode Completions of C_2

Given. The C_2 companion matrix and its spectrum:

$$A_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \lambda^2 - \lambda - 1 = 0, \qquad \mathrm{spec}(A_2) = \{\varphi,\ -1/\varphi\},$$

where $\varphi = (1 + \sqrt{5})/2$.

Completion 1 — Factor completion (sign/magnitude).

Every nonzero real eigenvalue has a unique factorization $\lambda = \mathrm{sign}(\lambda) \cdot |\lambda|$:

$$\varphi = (+1) \cdot \varphi, \qquad -1/\varphi = (-1) \cdot (1/\varphi).$$

Extract: signs $\{+1, -1\}$, magnitudes $\{\varphi, 1/\varphi\}$. Union:

$$\mathrm{spec}_1 = \{\varphi,\ 1/\varphi,\ +1,\ -1\}.$$

Reciprocal pairing ($\lambda_i \lambda_j = 1$): the pair $(\varphi, 1/\varphi)$ gives $\mathrm{sig}(1, 1)$; the self-reciprocal modes $+1$, -1 each give $\mathrm{sig}(1, 0)$. Total:

$$\mathrm{sig}_1 = (1, 1) + (1, 0) + (1, 0) = (3, 1). \qquad \blacksquare$$

Completion 2 — Symmetry completion (±-closure).

Close the magnitudes $\{\varphi,\ 1/\varphi\}$ under negation ($\lambda \to -\lambda$):

$$\mathrm{spec}_2 = \{\varphi,\ -\varphi,\ 1/\varphi,\ -1/\varphi\}.$$

Reciprocal pairing: $(\varphi,\ 1/\varphi)$ gives $\mathrm{sig}(1, 1)$; $(-\varphi,\ -1/\varphi)$ gives $\mathrm{sig}(1, 1)$. Total:

$$\mathrm{sig}_2 = (1, 1) + (1, 1) = (2, 2). \qquad \blacksquare$$

Note on terminology. Completion 1 is a unique real factorization, not a group closure. Completion 2 is a group-theoretic closure under negation. Both are canonically determined by $\mathrm{spec}(A_2)$ with no free choices.

Minimal parent.

Completion 1 gives $\mathrm{sig}(3, 1)$. Completion 2 gives $\mathrm{sig}(2, 2)$.

To contain $\mathrm{sig}(3, 1)$: $p \geq 3$, $q \geq 1$. To contain $\mathrm{sig}(2, 2)$: $p \geq 2$, $q \geq 2$. Together: $p \geq 3$, $q \geq 2$. Minimal: $\mathrm{sig}(3, 2)$, $\dim = 5$. Sufficiency: $\mathrm{sig}(3, 2)$ contains $\mathrm{sig}(3, 1)$ by dropping one negative direction, and $\mathrm{sig}(2, 2)$ by dropping one positive direction. The lower bound is attained.

This recovers the Minimal Parent Theorem (Proof 5) with both signatures now derived from $\mathrm{spec}(A_2)$ alone. $\blacksquare$

[No dependencies. Self-contained. Recovers Proofs 4 and 5.]